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Pronormal subgroups of a direct product of groups

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We give criteria to characterize abnormal, pronormal and locally pronormal subgroups of a direct product of two finite groups $A \times B$, under hypotheses of solvability for at least one of the factors, either A or B.

1. Introduction

The subgroups of a direct product of groups are well-understood. Direct products provide an elemental tool to construct groups and it is worthwhile to characterize the subgroups of a direct product which have other properties. This would elucidate whether a direct product would be a reasonable way to produce subgroups with one property but possibly not another.

The exact description of the subgroups of a direct product was given by Goursat (see [1] for a lucid description). Particular details are included in Section 2. The normal, subnormal, permutable, CAP, system permutable and normally embedded subgroups of a direct product have been studied in several articles [3,8–11,13,15]. For a survey article discussing various contributions to this research see [4].

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The purpose of this article is to add pronormality and related properties to the list of embedding properties that are well-understood in direct products.

We recall that a subgroup H of a group G is pronormal in G if H and H^g are conjugate in the join $\langle H, H^g\rangle$ for any element $g \in G$. This concept arises primarily from the basic properties of conjugacy and persistence of Sylow subgroups in finite groups and turns out to be an important property. It is the main subject of Section 6 in Chapter I of [7]. After Sylow subgroups, Hall subgroups and, more generally, injectors and projectors are pronormal in finite solvable groups. This is the reason that much of the background for this topic is in sources dealing primarily with solvable groups. However, no solvability is required by the definition, and that will be our first approach. We provide in Section 4 characterizations which require one of the direct factors to be solvable.

We deal also with abnormality, as a stronger but closely related property to pronormality. A subgroup H of a group G is abnormal in G if $g \in \langle H, H^g\rangle$ for all $g \in G$; equivalently, H is pronormal and self-normalizing in G. In this case, Carter subgroups in finite solvable groups are classical examples of abnormal subgroups. More generally, it is known that the normalizer of a pronormal subgroup is abnormal. This we will utilize. In fact, we first characterize the abnormal subgroups of a direct product in Section 3, with the assumption that one factor is solvable. With this hypothesis, abnormal subgroups of a direct product are exactly those subgroups which factor into a product of abnormal subgroups, one from each factor.

We investigate in Section 5 how classical pronormality and abnormality criteria for solvable groups by T.A. Peng [14] and G.J. Wood [16] extend in direct products. These criteria involve persistence in intermediate subgroups and we prove that only persistence into factorized intermediate subgroups is required to deduce pronormality or abnormality.

The topic of Section 6 is local pronormality. There are obvious implications from Section 4, dealing with pronormality, but, for a nilpotent subgroup, a weaker than expected condition will imply local pronormality in a direct product of two groups, one of which is solvable.

Section 2 will be used to establish the notation we shall use.

All groups considered here are assumed to be finite.

2. Preliminaries and notation

Let $G = A \times B$ be a direct product of groups A and B. We will adopt the notation of an internal direct product as much as it is sensible. So, G has two normal subgroups A and B with $A \cap B = 1$ and $G = AB$.

There are homomorphisms $\pi_A : G \to A$ by $\pi_A(ab) = a$ and $\pi_B : G \to B$ by $\pi_B(ab) = b$ for $a \in A$ and $b \in B$.

As mentioned in the Introduction, the characterization of the subgroups of a direct product dates back to Goursat (see [1]). It is easy to show that if $U \leqslant A \times B$, then $U \cap X \trianglelefteq \pi_X(U)$ for $X = A, B$, and there is an isomorphism $\sigma : \frac{\pi_A(U)}{U\cap A} \to \frac{\pi_B(U)}{U\cap B}$ with

$$U = \bigl\{ab\colon\ a \in \pi_A(U),\ b \in \pi_B(U),\ \bigl(a(A \cap U)\bigr)^{\sigma} = b(B \cap U)\bigr\}.$$

Conversely, if $I \trianglelefteq R \leqslant A$ and $J \trianglelefteq S \leqslant B$ such that there is an isomorphism $\sigma : R/I \to S/J$, then, for $U = \{rs\colon\ (rI)^{\sigma} = sJ\}$, it holds that $U \leqslant A \times B$ with $\pi_A(U) = R$, $U \cap A = I$, $\pi_B(U) = S$ and $U \cap B = J$.

Subsequent sections of this article are guided by insight into the "Goursat structure" of $N_G(U)$ where $U \leqslant G = A \times B$. For $X = A, B$, define $C_X = \{x \in X\colon\ [x, \pi_X(U)] \leqslant X \cap U\}$. It is clear that $C_X \leqslant N_X := N_X(\pi_X(U)) \cap N_X(X \cap U)$.

2.1. Proposition. *If $U \leqslant A \times B = G$ and C_X is as defined above for $X = A, B$, then $N_G(U) \cap X = C_X$.*

Proof. W.l.o.g. we prove the result for $X = A$. If $a \in N_G(U) \cap A$ and $x \in \pi_A(U)$, then there is $y \in B$ such that $xy \in U$. So $[a, xy] = [a, x] \in A \cap U$. Thus $a \in C_A$.

If $a \in C_A$ and $xy \in U$, where $x \in A$, $y \in B$, then $[a, xy] = [a, x] \in A \cap U$ by definition of C_A. But $[a, xy] = ((xy)^{-1})^a xy$. Thus $(xy)^a \in U$ and so $a \in N_G(U) \cap A$. □

To complete the description of $N_G(U)$ using the Goursat structure, one would look at subgroups R of $N_A \leqslant A$ and S of $N_B \leqslant B$ with an isomorphism $\Delta : R/C_A \to S/C_B$. The isomorphism here would need to be consistent with the isomorphism $\sigma : \pi_A(U)/U \cap A \to \pi_B(U)/U \cap B$. More exactly, the natural actions of R/C_A and S/C_B on $\pi_A(U)/U \cap A$ and $\pi_B(U)/U \cap B$, respectively, would need to be Δ-equivalent via σ; this is to say that

$$\left(\left(a(U \cap A)\right)^{rC_A}\right)^{\sigma} = \left(\left(a(U \cap A)\right)^{\sigma}\right)^{(rC_A)^{\Delta}}$$

for all $a \in \pi_A(U)$ and all $r \in R$. By choosing (R, S) maximal satisfying the mentioned properties, the projections of the normalizer would be located.

What we have said here generalizes the well known structure of normal subgroups of a direct product.

2.2. Corollary. *For $U \leqslant G = A \times B$, $U \trianglelefteq G$ if and only if $U \cap X \trianglelefteq X$ and $\pi_X(U)/U \cap X \leqslant Z(G/U \cap X)$ for $X = A, B$.*

3. Abnormal subgroups of a direct product

We give a very satisfying characterization of the abnormal subgroups in a direct product. Abnormal subgroups of non-solvable groups have been studied; [6] and [12] are some examples.

The latter part of Section 6 of Chapter 1 in [7] provides our basic tools for abnormality.

3.1. Definition. Let $U \leqslant G$. Then U is abnormal in G provided $g \in \langle U, U^g \rangle$ for each $g \in G$.

The first observation is straightforward to verify, but listed here for thoroughness sake.

3.2. Proposition. *Let X be an abnormal subgroup of A and Y an abnormal subgroup of B. Then $X \times Y$ is an abnormal subgroup of $A \times B$.*

3.3. Example. If S is any finite non-abelian simple group, and $U = \{(s, s)\colon\ s \in S\} \leqslant S \times S$ (viewing externally seems more natural here), then U is a non-normal maximal subgroup of $S \times S$. Since a maximal subgroup is either normal or abnormal (not both), U is abnormal in $S \times S$.

We notice that in the previous example, $\pi_S(U) = S$ is abnormal in S (for both components), U is abnormal in $S \times S$, but U is not a factorized subgroup of $S \times S$. We see next that under the additional hypothesis that one of the factors in the direct product is solvable, these conditions characterize abnormality.

We will freely apply the following facts which can be gleaned from [7]:

(i) If U is abnormal in G and $U \leqslant W \leqslant G$, then U is abnormal in W and W is abnormal in G.
(ii) If U is abnormal in G, then $N_G(U) = U$.
(iii) If U is abnormal in G and $\phi : G \to H$ is a homomorphism, then $\phi(U)$ is abnormal in $\phi(G)$.

First we establish a preliminary step.

3.4. Proposition. *Let $G = A \times B$ with either A or B solvable, and let M be a maximal subgroup of G with $N_G(M) = M$. Then $M = \pi_A(M) \times \pi_B(M)$ and either $\pi_A(M) = A$ or $\pi_B(M) = B$.*

Proof. We verify the contrapositive.

Suppose $\pi_A(M) \neq A \cap M$. Note that by Goursat this supposition is equivalent to supposing $\pi_B(M) \neq B \cap M$. Thus, without loss of generality we assume A is solvable.

Since M is maximal in G, and $M < \pi_A(M) \times \pi_B(M)$, it follows that $X = \pi_X(M)$ for $X = A, B$. Consequently, $X \cap M \trianglelefteqslant \pi_X(M) = X$ for $X = A, B$.

In fact, $A \cap M$ is a maximal normal subgroup of A. For if $A \cap M < W \trianglelefteqslant A$, then, since W is not contained in M, $G = MW$ and, consequently, $A = A \cap MW = (A \cap M)W = W$.

Then, since A is solvable, $A/A \cap M$ is abelian, and so also $B/B \cap M$ is abelian, and thus $M \trianglelefteqslant G$ which concludes the proof. □

3.5. Proposition. *Let $G = A \times B$ with one of A and B solvable and suppose U is abnormal in G. Then $U = \pi_A(U) \times \pi_B(U)$ and $\pi_X(U)$ is abnormal in X for $X = A, B$.*

Proof. For $X = A, B$, $\pi_X : G \to X$ is a surjective homomorphism and so $\pi_X(U)$ is abnormal in X.

Using induction on $|G|$ to see that $U = \pi_A(U) \times \pi_B(U)$, let M be a maximal subgroup of G containing U. Since M is abnormal in G, $M = \pi_A(M) \times \pi_B(M)$ from Proposition 3.4. Also U is abnormal in M with $|M| < |G|$. It follows that $U = \pi_{\pi_A(M)}(U) \times \pi_{\pi_B(M)}(U) = \pi_A(U) \times \pi_B(U)$. □

We isolate one very definitive conclusion.

3.6. Proposition. *Let $G = A \times B$ where either A or B is solvable and suppose $U \leqslant G$. Then U is abnormal in G if and only if $\pi_X(U)$ is abnormal in X for $X = A, B$, and $U = \pi_A(U) \times \pi_B(U)$.*

The abnormal subgroups in finite simple groups have only been cursorily studied [6,12]. To classify them in a direct product of simple groups eludes us. From [12] one should be aware that it is not sufficient to consider subgroups U in which $U \leqslant W$ implies $W = N_G(W)$. (This condition is known to be sufficient to guarantee the abnormality of a subgroup U when the group G is solvable [14].) Our comment in Section 2 about normalizers would have even a lessened effect.

4. Pronormal subgroups of a direct product

The concept of a pronormal subgroup was introduced by P. Hall in his lectures at Cambridge University. Section 6 of Chapter I in [7] provides a thorough, interesting account of pronormality.

4.1. Definition. If $U \leqslant G$, then U is pronormal in G provided that for each $g \in G$, there is $x \in \langle U, U^g \rangle$ such that $U^g = U^x$.

Certainly both normal and abnormal subgroups are pronormal. Less obvious examples are the Sylow subgroups of a normal subgroup. As mentioned in the Introduction, for solvable groups, both projectors and injectors are pronormal because of their persistence properties. While pronormality is most commonly studied in solvable groups, we will follow at a first step our direction from the previous section by only requiring that one component of the direct product is solvable. Nevertheless we consider afterwards in Section 5 classical pronormality, and also abnormality, criteria for solvable groups, and study their behavior when particularizing to direct products of solvable groups.

For convenience we will cite some facts about pronormality that will be used in our main result. The reader should note that solvability is not used in their proofs contained in [7, I.6.3 and I.6.4].

4.2. Lemma. *Let G be a group. Then*:

(i) *If U is pronormal in G, then $N_G(U)$ is abnormal in G.*
(ii) *If $U \leqslant G$, then U is both subnormal and pronormal in G if and only if $U \trianglelefteqslant G$.*
(iii) *If $U \leqslant K \trianglelefteqslant G$ and U is pronormal in G, then $G = N_G(U)K$.*
(iv) *If $N \trianglelefteqslant G$ and U is pronormal in G, then UN is pronormal in G; furthermore, $N_G(UN) = N_G(U)N$.*
(v) *Suppose $\phi : G \to H$ is a group epimorphism. Then* (a) *if U is pronormal in G, then $\phi(U)$ is pronormal in H, and* (b) *if W is pronormal in H, then $\phi^{-1}(W)$ is pronormal in G.*

(vi) (*Gaschütz*) *Let $U \leqslant G$. Then U is pronormal in G if and only if for some $N \trianglelefteq G$, U is pronormal in $N_G(UN)$ and UN is pronormal in G.*

4.3. Proposition. *Let $U \leqslant G = A \times B$. Assume that the following conditions hold*:

(i) *$\pi_X(U)$ is pronormal in X for $X = A, B$;*
(ii) *$N_G(U) = N_A(\pi_A(U)) \times N_B(\pi_B(U))$.*

Then U is pronormal in G.

Proof. Suppose this result is false. Let G be a group of minimal order possessing a non-pronormal subgroup U satisfying both (i) and (ii) of the hypothesis. Moreover suppose that U is chosen of maximal order among those non-pronormal subgroups of G which satisfy both (i) and (ii).

Let $N \trianglelefteq A$. Then $N \trianglelefteq G$, and $\pi_A(UN) = \pi_A(U)N$ is pronormal in A by (i) of the hypothesis in conjunction with Lemma 4.2(iv). Moreover, since $N \leqslant A$, $\pi_B(UN) = \pi_B(U)$ is pronormal in B. Hence UN satisfies (i) of the hypothesis.

Moreover, since $\pi_A(U)$ is pronormal in A, using Lemma 4.2(iv) it follows that $N_A(\pi_A(U)N) = N_A(\pi_A(U))N$. Thus

$$\begin{aligned} N_G(UN) &\leqslant N_A\big(\pi_A(UN)\big) \times N_B\big(\pi_B(UN)\big) \\ &= N_A\big(\pi_A(U)N\big) \times N_B\big(\pi_B(U)\big) \\ &= N_A\big(\pi_A(U)\big)N \times N_B\big(\pi_B(U)\big) \\ &= \big(N_A\big(\pi_A(U)\big) \times N_B\big(\pi_B(U)\big)\big)N = N_G(U)N \leqslant N_G(UN) \end{aligned}$$

by hypothesis (ii) applied to U. Therefore UN satisfies (ii) of the hypothesis.

For $N \trianglelefteq A$ either $N \leqslant U$ or $UN \neq U$. If $N \leqslant U$ for all $N \trianglelefteq A$, then $A \subseteq U$ and so $U = A \times \pi_B(U)$ and easily U is pronormal in G, contrary to choice.

Hence there is $N \trianglelefteq A$ with $UN \neq U$. Thus UN is pronormal in G.

Now either $UN \trianglelefteq G$ or $N_G(UN) < G$. Suppose $N_G(UN) < G$. From the fact that UN satisfies (ii), $N_G(UN) = A \cap N_G(UN) \times B \cap N_G(UN)$ is a direct product and U satisfies (i) and (ii) in $N_G(UN)$. Consequently, U is pronormal in $N_G(UN)$. However, Gaschütz's result, Lemma 4.2(vi), would then imply that U is pronormal in G.

Thus, it must be that $UN \trianglelefteq G$. Hence, $\pi_B(UN) = \pi_B(U) \trianglelefteq B$.

This argument is symmetric with respect to A and B and so one should conclude also that $\pi_A(U) \trianglelefteq A$.

But (ii) gives $N_G(U) = N_A(\pi_A(U)) \times N_B(\pi_B(U)) = A \times B$. That is, $U \trianglelefteq G$, which is again contrary to choice.

Thus there are no counterexamples and the result is proven. $\square$

As Example 3.3 dictates, in order to hope that the conditions (i) and (ii) in Proposition 4.3 would characterize the pronormal subgroups of a direct product, an extra hypothesis is required.

4.4. Proposition. *Let $U \leqslant G = A \times B$ with one of A or B solvable. Then U is pronormal in G if and only if $\pi_X(U)$ is pronormal in X for $X = A, B$, and $N_G(U) = N_A(\pi_A(U)) \times N_B(\pi_B(U))$.*

Proof. Assume that U is pronormal in G. From Lemma 4.2(v), it follows that $\pi_X(U)$ is pronormal in X for $X = A, B$. From Lemma 4.2(i), $N_G(U)$ is abnormal in G. By Proposition 3.5, $N_G(U) = A_U \times B_U$ where $X_U = \pi_X(N_G(U)) = X \cap N_G(U)$ for $X = A, B$. Note that $X_U \cap U = X \cap U$ and $\pi_X(U) \leqslant X_U$. Utilizing Corollary 2.2, since $U \trianglelefteq A_U \times B_U$, it follows that $\pi_X(U)/X \cap U$ is abelian, and so $U \trianglelefteq \pi_A(U) \times \pi_B(U)$. Thus $U \trianglelefteq\trianglelefteq N_G(\pi_A(U) \times \pi_B(U)) = N_A(\pi_A(U)) \times N_B(\pi_B(U))$. From Lemma 4.2(ii), we conclude that $N_G(U) \geqslant N_A(\pi_A(U)) \times N_B(\pi_B(U))$. Since $\pi_X(N_G(U)) \leqslant N_X(\pi_X(U))$ for $X = A, B$, it follows that $N_G(U) = N_A(\pi_A(U)) \times N_B(\pi_B(U))$.

The converse follows by Proposition 4.3. $\square$

We point out the following fact about the structure of pronormal subgroups in direct products, appearing in the proof of the previous result.

4.5. Corollary. *Let $U \leqslant A \times B = G$ with one of A or B solvable. If U is pronormal in G, then $\pi_X(U)/(U \cap X)$ is abelian for $X = A, B$. In particular U is normal in $\pi_A(U) \times \pi_B(U)$.*

4.6. Remark. (1) The converse of Corollary 4.5 is not true. It is enough to consider $A = B \cong \mathrm{Sym}(3)$ the symmetric group on three letters and $G = A \times B$. Let $S = O_3(A) = O_3(B)$ and $D = \{(x, x) \in G\colon\ x \in S\}$. Then $\pi_X(D)/(D \cap X) = \pi_X(D) = S$ is abelian, for $X = A, B$, but D is not pronormal in G.

(2) If neither of the factors A nor B of the group $G = A \times B$ is solvable, Corollary 4.5 is not true. Example 3.3 shows this.

Recall from Proposition 2.1 that for $U \leqslant A \times B = G$, $N_G(U) \cap X = C_X = \{x \in X\colon\ [x, \pi_X(U)] \leqslant U \cap X\}$ for $X = A, B$.

4.7. Corollary. *Let $U \leqslant A \times B = G$ with one of A or B solvable. Then U is pronormal in G if and only if $\pi_X(U)$ is pronormal in X and $N_X(\pi_X(U)) \leqslant C_X$ for $X = A, B$.*

Proof. By Proposition 4.4, it suffices to show that $N_G(U) = N_A(\pi_A(U)) \times N_B(\pi_B(U))$ if and only if $N_X(\pi_X(U)) \leqslant C_X$ for $X = A, B$.

Assume first that $N_X(\pi_X(U)) \leqslant C_X$ for $X = A, B$. Note from Proposition 2.1,

$$\begin{aligned} N_A\big(\pi_A(U)\big) \times N_B\big(\pi_B(U)\big) &\leqslant N_G(U) \\ &\leqslant \pi_A\big(N_G(U)\big) \times \pi_B\big(N_G(U)\big) \\ &\leqslant N_A\big(\pi_A(U)\big) \times N_B\big(\pi_B(U)\big). \end{aligned}$$

Conversely, if $N_G(U) = N_A(\pi_A(U)) \times N_B(\pi_B(U))$, then $U \trianglelefteqslant N_A(\pi_A(U)) \times N_B(\pi_B(U))$ and so $\frac{\pi_X(U)}{U \cap N_X(\pi_X(U))} \leqslant Z(\frac{N_X(\pi_X(U))}{U \cap N_X(\pi_X(U))})$ for $X = A, B$, by Corollary 2.2. Since $U \cap X \geqslant U \cap N_X(\pi_X(U))$ for $X = A, B$, it follows that $N_X(\pi_X(U)) \leqslant C_X$. $\square$

4.8. Remark *(Construction of pronormal subgroups of a direct product of groups $A \times B$ with either A or B solvable). First choose R pronormal in A, S pronormal in B. Consider*

$$I_A = \big\{I\colon\ \big[R, N_A(R)\big] \leqslant I \leqslant R\big\}$$

and

$$I_B = \big\{J\colon\ \big[S, N_B(S)\big] \leqslant J \leqslant S\big\}.$$

Then for $I \in I_A$, $J \in I_B$ such that $R/I \cong S/J$, one can construct, as in Section 2, a subgroup U of $A \times B$, which by Corollary 4.7 is pronormal in $A \times B$, such that $\pi_A(U) = R$, $\pi_B(U) = S$, $U \cap A = I$ and $U \cap B = J$.

Corollary 4.7 implies that any pronormal subgroup of $A \times B$ is of this type.

5. Characterizations of pronormal and abnormal subgroups in direct products of solvable groups

In this section we investigate how two of the well-known characterizations of pronormality and abnormality in solvable groups can be modified in a direct product. The first of these is due to T.A. Peng [14]. We recall the concept of weak Frattini argument from [5].

5.1. Definition. A subgroup X of a group Y is said to *satisfy the weak Frattini argument* in Y if $Y = KN_Y(X)$ whenever $X \leqslant K \trianglelefteq Y$. In this case we will write $X \in WFA(Y)$.

5.2. Proposition. *(See Peng [14].) If X is a subgroup of a solvable group Y, then X is pronormal in Y if and only if $X \in WFA(L)$, whenever $X \leqslant L \leqslant Y$.*

Feldman's example in [12] shows that solvability is required in Peng's result. It is always true that a pronormal subgroup satisfies the weak Frattini argument.

For direct products this characterization of pronormal subgroups extends in the following way, by considering only intermediate subgroups which are factorized.

5.3. Lemma. *A subgroup H of a solvable group $G = A \times B$ is pronormal in G if and only if H satisfies the following conditions*:

(i) $H \trianglelefteq \pi_A(H) \times \pi_B(H)$;
(ii) *Whenever $H \leqslant K \trianglelefteq L \leqslant G$ such that $K = \pi_A(K) \times \pi_B(K)$ and $L = \pi_A(L) \times \pi_B(L)$, then $L = N_L(H)K$.*

Proof. If H is pronormal in G it is known that conditions (i) and (ii) are satisfied.

For the converse, if either $A = 1$ or $B = 1$, then the result follows from Proposition 5.2. Assume that $A \neq 1$ and $B \neq 1$. We argue by induction on $|G|$. Let $1 \neq N \trianglelefteq A$. A straightforward computation shows that HN/N satisfies conditions (i) and (ii) in $G/N = A/N \times BN/N$. Then from the inductive hypothesis and Lemma 4.2(v), it follows that HN is pronormal in G.

Assume that $A \times N_B(\pi_B(H)) < A \times B$. We notice that $H \leqslant \pi_A(H) \times \pi_B(H) \leqslant A \times N_B(\pi_B(H))$ and H satisfies (i) and (ii) with respect to $A \times N_B(\pi_B(H))$. By the inductive hypothesis H is pronormal in $A \times N_B(\pi_B(H))$. But $H \leqslant N_G(HN) \leqslant N_G(\pi_B(HN)) = N_G(\pi_B(H)) = A \times N_B(\pi_B(H))$, so H is pronormal in $N_G(HN)$ and we are done by Lemma 4.2(vi). We may now assume that $N_B(\pi_B(H)) = B$, that is, $\pi_B(H) \trianglelefteq B$, and analogously that $\pi_A(H) \trianglelefteq A$. Conditions (i) and (ii) imply that $G = (\pi_A(H) \times \pi_B(H))N_G(H) = N_G(H)$ which concludes the proof. $\square$

5.4. Lemma. *Assume that $A \cong B$ and let H be a main diagonal subgroup of $G = A \times B$, i.e., $\pi_X(H) = X$ and $H \cap X = 1$ for $X = A, B$. Then $N_G(H) = HZ(G)$.*

Proof. By Proposition 2.1 we have that $N_G(H) \cap X = C_X(X) = Z(X)$ for $X = A, B$. Moreover, since H is a main diagonal subgroup of G, it follows that $X \cong H$ and $Z(H) = H \cap Z(G)$, which implies that $|X/Z(X)| = |H/H \cap Z(G)|$ for $X = A, B$. So we notice that $|N_G(H)/Z(G)| = |N_G(H)/(Z(A) \times Z(B))| = |\pi_X(N_G(H))/Z(X)| \geqslant |\pi_X(H)/Z(X)| = |X/Z(X)| = |H/H \cap Z(G)| = |HZ(G)/Z(G)|$, for $X = A, B$. Then $|N_G(H)| = |HZ(G)|$ and so $N_G(H) = HZ(G)$. $\square$

5.5. Proposition. *A subgroup H of the solvable group $G = A \times B$ is pronormal in G if and only if $H \in WFA(L)$, whenever $H \leqslant L \leqslant G$ such that $L = \pi_A(L) \times \pi_B(L)$.*

Proof. Assume that $H \in WFA(L)$, whenever $H \leqslant L \leqslant G$ such that $L = \pi_A(L) \times \pi_B(L)$ and prove that H is pronormal in G. The other implication is known. We argue by induction on $|G|$. Assume $\pi_A(H) \times B$ is a proper subgroup of G. Then by the inductive hypothesis H is pronormal in $\pi_A(H) \times B$ which implies $H \trianglelefteq \pi_A(H) \times \pi_B(H)$ by Corollary 4.5. Hence H is pronormal in G by Lemma 5.3. So we may assume that $\pi_A(H) = A$ and analogously $\pi_B(H) = B$. In particular, $H \cap X \trianglelefteq X$ for $X = A, B$. If $H \cap X \neq 1$ for some $X \in \{A, B\}$, then $H/H \cap X$ is pronormal in $G/H \cap X$ by the inductive hypothesis and so H is pronormal in G by Lemma 4.2(v). Therefore we may also assume $H \cap A = H \cap B = 1$, which means that $A = \pi_A(H)/(H \cap A) \cong \pi_B(H)/(H \cap B) = B$ and H is a main diagonal subgroup of G.

Let α be an isomorphism from A onto B such that $H = \{aa^\alpha \colon a \in A\}$.

If G is abelian it is clear that H is pronormal in G. Otherwise there exists a maximal normal subgroup M of A containing $Z(A)$. Let $N = M^\alpha$ and consider $A = \dot{\bigcup}_{t\in T} Mt$, T a right transversal of M in A. Then $K := (M \times N)\{tt^\alpha \colon t \in T\}$ is a normal proper subgroup of G and $H \leqslant K$. By the hypothesis, $G = KN_G(H)$. But $Z(G) \leqslant K$ and so by Lemma 5.4 we have $KN_G(H) = KZ(G)H = K = G$, a contradiction which concludes the proof. $\square$

5.6. Remark. From Lemma 5.3 one might wonder whether for a subgroup H of a solvable group $G = A \times B$, H is pronormal in G if and only if $L = N_L(H)K$ whenever $H \leqslant K \trianglelefteq L \leqslant G$ such that $K = \pi_A(K) \times \pi_B(K)$ and $L = \pi_A(L) \times \pi_B(L)$. This is not so. To see this we consider $G = A \times B$ with $A = B \cong \mathrm{Sym}(3)$ and $H = \{(x, x) \in G \colon x \in A = B\}$. Then H is not pronormal in G but it satisfies the mentioned condition.

We consider now the following pronormality and abnormality criteria for solvable groups due to G.J. Wood [16]. These criteria are defined by conditions that are persistent in intermediate subgroups.

5.7. Proposition. *(See Wood [16].) If X is a subgroup of a solvable group Y, then the following are equivalent*:

1. *X is pronormal in Y;*
2. *$N_L(X)$ is abnormal in L, whenever $X \leqslant L \leqslant Y$;*
3. *$N_L(X)$ contains some system normalizer of L, whenever $X \leqslant L \leqslant Y$.*

5.8. Proposition. *(See Wood [16].) Let X be a subgroup of a solvable group Y. For each subgroup L of Y choose D_L some system normalizer of L. If X is pronormal in $\langle X, D_L\rangle$ for each subgroup L with $X \leqslant L \leqslant Y$, then X is pronormal in Y.*

5.9. Proposition. *(See Wood [16].) For a subgroup X of a solvable group Y, the following are equivalent*:

1. *X is abnormal in Y;*
2. *X contains some system normalizer of L, whenever $X \leqslant L \leqslant Y$;*
3. *whenever $X < L \leqslant Y$, then there exists D_L, some system normalizer of L, with $\langle X, D_L\rangle < L$.*

For direct products we again show the conditions for pronormality and for abnormality are sufficient considering only intermediate factorized subgroups.

5.10. Lemma. *A subgroup H of a solvable group $G = A \times B$ is pronormal in G if and only if H satisfies the following conditions*:

(i) *$H \trianglelefteq \pi_A(H) \times \pi_B(H)$;*
(ii) *$N_L(H)$ contains some system normalizer of L, whenever $H \leqslant L \leqslant G$ such that $L = \pi_A(L) \times \pi_B(L)$.*

Proof. From Corollary 4.5 and Proposition 5.7, a pronormal subgroup satisfies (i) and (ii). For the converse, we argue as in Lemma 5.3. In particular, we argue by induction on $|G|$ to prove that conditions (i) and (ii) imply that H is pronormal in G. As in the proof of Lemma 5.3 with the suitable changes, we may assume that $A \neq 1$, $B \neq 1$ and $\pi_A(H), \pi_B(H) \trianglelefteq G$. By (i) we may also assume w.l.o.g. that $\pi_A(H)$ is a proper subgroup of A. Then let M be a maximal normal subgroup of A containing $\pi_A(H)$. By inductive hypothesis we have now that H is pronormal in $M \times B$. Moreover $H \trianglelefteq \pi_A(H) \times \pi_B(H) \trianglelefteq G$, which implies that $H \trianglelefteq M \times B$. On the other hand, by (ii) there exists a system normalizer D of G such that $D \leqslant N_G(H)$. But $G = (M \times B)D$ by [7, Theorem I.5.6] and so $H \trianglelefteq G$, which concludes the proof. $\square$

5.11. Proposition. *For a subgroup H of a solvable group $G = A \times B$, the following are equivalent*:

1. *H is pronormal in G;*
2. *$N_L(H)$ is abnormal in L, whenever $H \leqslant L \leqslant G$ such that $L = \pi_A(L) \times \pi_B(L)$;*
3. *$N_L(H)$ contains some system normalizer of L, whenever $H \leqslant L \leqslant G$ such that $L = \pi_A(L) \times \pi_B(L)$.*

Proof. That Condition 1 implies Condition 2 and that Condition 2 implies Condition 3 follow from Proposition 5.7. Then we only need to prove that Condition 3 implies Condition 1. Assume that $N_L(H)$ contains some system normalizer of L, whenever $H \leqslant L \leqslant G$ such that $L = \pi_A(L) \times \pi_B(L)$, and prove that H is pronormal in G. From Lemma 5.10 and arguing as in Proposition 5.5, we may conclude analogously that $H \cap X = 1$ and $\pi_X(H) = X$, for $X = A, B$; in particular, $A \cong B$ and H is a main diagonal subgroup of G. By hypothesis there exists a system normalizer D of G such that $D \leqslant N_G(H)$. But $D = D_A \times D_B$, where D_X is a system normalizer of X for $X = A, B$. Then $D_X \leqslant N_G(H) \cap X = C_X(X) = Z(X)$ for $X = A, B$, by Proposition 2.1. This means that $D \leqslant Z(G)$ and so $G = \langle D^G \rangle \leqslant Z(G)$ by [7, Theorem I.5.9(a)], that is, G is abelian and we are done. □

5.12. Proposition. *Let H be a subgroup of a solvable group $G = A \times B$. For each subgroup L of G such that $H \leqslant L = \pi_A(L) \times \pi_B(L) \leqslant G$ choose D_L some system normalizer of L. If H is pronormal in $\langle H, D_L \rangle$ for each subgroup L such that $H \leqslant L = \pi_A(L) \times \pi_B(L) \leqslant G$, then H is pronormal in G.*

Proof. By Proposition 5.11 and arguing by induction on the order of G, it is enough to prove that $N_G(H)$ contains some system normalizer of G. We let $C := \langle H, D_G \rangle$ with D_G the chosen system normalizer of G. If $C = G$, then H is pronormal in G by hypothesis. Otherwise there exists a maximal subgroup M of G containing C. Since $D_G \leqslant M$ we have that M is abnormal in G but then $M = \pi_A(M) \times \pi_B(M)$ by Proposition 3.6. By the inductive hypothesis $N_M(H)$ contains some system normalizer of M. But a system normalizer of M contains a system normalizer of G which concludes the proof. □

5.13. Proposition. *For a subgroup H of a solvable group $G = A \times B$, the following are equivalent*:

1. *H is abnormal in G;*
2. *H contains some system normalizer of L, whenever $H \leqslant L = \pi_A(L) \times \pi_B(L) \leqslant G$;*
3. *whenever $H < L = \pi_A(L) \times \pi_B(L) \leqslant G$, then there exists D_L, some system normalizer of L, with $\langle H, D_L \rangle < L$.*

Proof. Again we know that Condition 1 implies Condition 2 and that Condition 2 implies Condition 3. If Condition 3 holds we can assume that in particular that $\langle H, D_G \rangle < G$ with D_G a system normalizer of G. Arguing by induction on the order of G we have also that H is abnormal in any subgroup L such that $H \leqslant L = \pi_A(L) \times \pi_B(L) < G$. Let M be a maximal subgroup of G containing $D_G \leqslant \langle H, D_G \rangle$. Then M is abnormal in G and so $M = \pi_A(M) \times \pi_B(M)$ by Proposition 3.6. Therefore H is abnormal in M which implies that $H = \pi_A(H) \times \pi_B(H)$ and $\pi_X(H)$ is abnormal in $\pi_X(M)$ for $X = A, B$. W.l.o.g. we may assume that $\pi_A(M) = A$ and in particular $\pi_A(H)$ is abnormal in A. Moreover $\pi_B(H)$ contains a system normalizer of B because it contains a system normalizer of $\pi_B(M)$ which is abnormal in B. We can deduce now that $\pi_B(H)$ is abnormal in B which implies finally that H is abnormal in G by Proposition 3.6. □

6. Locally pronormal subgroups of a direct product

A subgroup U of a group G is called *locally pronormal* in G provided that for each prime p, a Sylow p-subgroup of U is pronormal in G. In an arbitrary group there is no necessary containment between the set of pronormal subgroups and the set of locally pronormal subgroups. However, for a solvable group G, a locally pronormal subgroup of G is pronormal in G. There are examples, including Example 6.4(a), of pronormal subgroups that are not locally pronormal.

Since local pronormality directly involves the Sylow subgroups, it would be natural to consider the relation between locally pronormal and normally embedded subgroups. Certainly normally embedded subgroups are locally pronormal. Much about these properties and the connections between them can be found in [7, Sections 6 and 7]. In [3] several characterizations of normally embedded subgroups in a direct product were found.

Here we seek to investigate locally pronormal subgroups of a direct product. Of course there is an immediate result from Section 4.

6.1. Proposition. *Let $U \leqslant G = A \times B$ where one of A or B is solvable. Then U is locally pronormal in G if and only if for each prime p and $P \in \mathrm{Syl}_p(U)$, $\pi_X(P)$ is pronormal in X for $X = A, B$, and $N_G(P) = N_A(\pi_A(P)) \times N_B(\pi_B(P))$.*

Our hope was that the structure imposed by the direct product might make possible a characterization that looks weaker. Without additional structure limitations on U we were unsuccessful. Thus we require U to be nilpotent. There are other situations in the literature where different embedding properties, when applied to a nilpotent subgroup, coincide; see for instance [2] (also [7, Problem 4, p. 553]).

We consider the possibility that if $U \leqslant G = A \times B$ is pronormal in G and $\pi_X(U)$ is locally pronormal in X for $X = A, B$, then U is locally pronormal in G. Our Example 6.4(b) will show this possibility is not valid in general, even if G is solvable. Proposition 6.2 will verify the desired result in case U is nilpotent.

6.2. Proposition. *Let $U \leqslant G = A \times B$ where one of A or B is solvable and U is nilpotent. If U is pronormal in G and $\pi_X(U)$ is locally pronormal in X for $X = A, B$, then U is locally pronormal in G.*

Proof. Suppose the proposition is false, that G is a counterexample of minimal order and $P \in \mathrm{Syl}_p(U)$, for some prime p, such that P is not pronormal in G. Then:

(1) $\pi_X(P) \trianglelefteq X$ for $X = A, B$, and consequently $P \trianglelefteq\trianglelefteq G$.

To see this, suppose either $N_A(\pi_A(P)) < A$ or $N_B(\pi_B(P)) < B$. Then $W := N_A(\pi_A(P)) \times N_B(\pi_B(P)) < G$, and $U \leqslant N_G(P) \leqslant W$ satisfies the hypotheses, and so P is pronormal in W. Since P is a p-group we deduce $P \trianglelefteq\trianglelefteq \pi_A(P) \times \pi_B(P) \trianglelefteq W$, which implies $P \trianglelefteq W$ and so $N_G(P) = W = N_A(\pi_A(P)) \times N_B(\pi_A(P))$. Proposition 4.4 would imply that P is pronormal in G, contrary to choice.

So $\pi_X(P) \trianglelefteq X$ for $X = A, B$. It easily follows that $P \trianglelefteq\trianglelefteq G$ as P is a p-group.

(2) If $N \trianglelefteq G$ with $N \neq 1$, and $N \leqslant A$ or $N \leqslant B$, then $PN \trianglelefteq G$.

The cases $N \leqslant A$ and $N \leqslant B$ are symmetric. We argue the case when $N \leqslant A$.

Then $UN/N \leqslant G/N \cong A/N \times B$, UN/N is nilpotent, UN/N is pronormal in G/N and the hypothesis on the projections is satisfied. So $PN/N \in \mathrm{Syl}_p(UN/N)$ and PN/N is pronormal in G/N. It is also subnormal in G/N by (1) and so $PN \trianglelefteq G$.

(3) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, there exists a non-trivial normal p'-subgroup N of either A or B. Then P is a subnormal Sylow p-subgroup of $PN \trianglelefteq G$. P is characteristic in PN and consequently is normal in G, contrary to choice.

(4) Either $\pi_A(U) = A$ or $\pi_B(U) = B$.

Assume that $\pi_A(U) < A$. Since $U \leqslant \pi_A(U) \times B$ and satisfies the hypotheses of the statement there, the choice of G implies that P is pronormal in $\pi_A(U) \times B$ and so $\pi_A(U) \times B \leqslant N_G(P)$. If $\pi_B(U) < B$, the same argument would yield that $A \times \pi_B(U) \leqslant N_G(P)$ and, hence, $P \trianglelefteq G$. This is contrary to choice.

(5) Final contradiction.

We suppose $\pi_B(U) = B$. However U is nilpotent and, from (3), $O_{p'}(B) = 1$ and so $\pi_B(U) = B$ is a p-group.

Since U is pronormal in G, we have $P \neq U$ and so there is $1 \neq Q \in \mathrm{Syl}_q(U)$ for $q \neq p$. Then $\pi_B(Q) = 1$ and so $Q = \pi_A(Q) \in \mathrm{Syl}_q(\pi_A(U))$. Thus Q is pronormal in A. From (1) $\pi_A(P) \trianglelefteq A$ and so $C_A(\pi_A(P)) \trianglelefteq A$. Since U is nilpotent, $Q \leqslant C_A(\pi_A(P))$. From Lemma 4.2(iii), $A = C_A(\pi_A(P))N_A(Q)$.

Note that since $O_{p'}(G) = 1$, $N_A(Q) < A$. Also, since U is nilpotent, $U \leqslant N_A(Q) \times B$ and the hypotheses for the statement are valid there. Hence $P \trianglelefteq N_A(Q) \times B$. Since $C_A(\pi_A(P)) \leqslant N_G(P)$, it follows $C_A(\pi_A(P))(N_A(Q) \times B) = A \times B \leqslant N_G(P)$.

This final contradiction implies that there are no counterexamples. $\square$

6.3. Corollary. *Let $G = A \times B$ be a solvable group and $U \leqslant G$ with U nilpotent. Then U is locally pronormal in G if and only if U is pronormal in G and $\pi_X(U)$ is locally pronormal in X for $X = A, B$.*

6.4. Examples.

(a) [7, Problem 15, p. 250] A cyclic pronormal subgroup of G need not be locally pronormal in G even if G is metabelian.
(b) If $U \leqslant G = A \times B$ and U is not nilpotent, it may be that U is not locally pronormal in G even though U is pronormal in G, $\pi_X(U)$ is locally pronormal in X for $X = A, B$, and G' is nilpotent.

Proof. Let $P = \langle a, b\colon\ a^3 = b^3 = [a,b]^3 = 1 = [a,b,b] = [a,b,a]\rangle$ the extraspecial group of order 27 and exponent 3.

(a) It is easily checked that $\beta : {\substack{a \to a^{-1} \\ b \to b}}$ induces an automorphism of P. Let $G = P\langle\beta\rangle$, the corresponding semidirect product. Let $E = \langle b, \beta\rangle$. Then E is a cyclic subgroup of G of order 6. Since $G' = \langle [a,b], a\rangle$ and $G = G'E$, one can see that E is a Carter subgroup of G. Hence, E is pronormal in G. However, $\langle b\rangle$ is subnormal but not normal in P and so E is not locally pronormal in G.

(b) For this example we use the notation of an external direct product. In P, let $z = [a,b]$.

Let $V \leqslant P \times P$, $V = \langle (a,1), (1,a), (z,1), (1,z), (b,b^{-1})\rangle$. Note that $P \times P$ has an automorphism $\alpha : (x,y) \to (y,x)$ for $x, y \in P$, and V is invariant under α.

Let $A = V\langle\alpha\rangle$, the corresponding semidirect product. Let $D = \langle (z,z), (a,a)\rangle$ and $\bar{D} = \langle (z,z^{-1}), (a,a^{-1})\rangle$. Both D and $D\bar{D}$ are subgroups of A normalized by α. D and $\bar{D}$ are isomorphic to $C_3 \times C_3$. Also, $\alpha \in C_A(D)$.

Let $U_A = D\bar{D}\langle\alpha\rangle \leqslant A$. Note $D\bar{D} \in \mathrm{Syl}_3(U_A)$ and $D\bar{D} \trianglelefteq A$. Also $\langle\alpha\rangle \in \mathrm{Syl}_2(A)$ and so U_A is locally pronormal in A. One can see that U_A is a maximal subgroup of A but $(b,b^{-1}) \notin N_A(U_A)$. Thus $U_A = N_A(U_A)$.

Now $\bar{D} \trianglelefteq U_A$ and $U_A \leqslant C_A(U_A/\bar{D})$ with $U_A/\bar{D} \cong C_3 \times C_3 \times C_2$.

Let $\theta : U_A \to C_3 \times C_3 \times C_2 = B$ be an epimorphism with $\ker\theta = \bar{D}$. Set $U = \{(x,y) \in A \times B\colon\ x \in U_A$ and $x^\theta = y\}$. By Remark 4.8, U is pronormal in $A \times B$. $\pi_A(U) = U_A$ is locally pronormal in A from above. Certainly $\pi_B(U) = B$ is locally pronormal in B.

However, $D\bar{D} = \pi_A(T)$, for one $T \in \mathrm{Syl}_3(U)$, and $T \cap A = \bar{D}$. Now $A = N_A(D\bar{D})$ but $A \nleqslant N_A(\bar{D})$ since $(b,b^{-1}) \notin N_A(\bar{D})$. Again from Remark 4.8, T is not pronormal in $A \times B$. Hence U is not locally pronormal in $A \times B$. $\square$

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